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Numerical solution of nonlinear partial differential equations with the Tau method

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Abstract: The ability of a recent formulation of the Tau method of Ortiz and Samara to give approximate solutions of a high accuracy of linear partial differential equations with variable coefficients is used to produce numerical solutions of nonlinear partial differential equations. Examples given in this paper show that even for relatively low degrees, Tau approximations give a high degree of accuracy. Besides, the approximate solution and all its derivatives are continuous in the domain.

Keywords: Tau method, nonlinear partial differential equations.

1. Introduction

In a recent paper Ortiz and Samara [11] proposed a reformulation of Ortiz' [7] approach to the Tau method which simplifies considerably the process of construction of the Tau approximant.

The same technique has been used by these authors [12] for the approximate solution of linear partial differential equations with the Tau method. As their approach is applicable to partial differential equations with variable (polynomial or rational polynomial) coefficients, it is possible to use for nonlinear partial differential equations a linearization technique introduced by Ortiz [9] for the numerical treatment of nonlinear *ordinary* differential equations.

This problem is discussed in this paper, where we report on numerical experiments conducted on three types of nonlinear partial differential equations.

The types considered are: (i) an elliptic boundary value problem with a cubic term, (ii) a parabolic problem with a quadratic term, and (iii) an hyperbolic problem with a quadratic term.

In all three cases the exact solution is known, so that comparisons are possible. We report results of a very high accuracy even when we have used Tau approximations of a relatively low degree.

As the Tau approximation is a bivariate polynomial, it can be differentiated freely in the domain of solution.

2. The Tau method for nonlinear partial differential equations

The basic principle of the Tau method (see Lanczos [3]) is the replacement of the original problem by another one, which we call the *Tau problem*, defined by the same differential equation, but with an appended polynomial term in the right hand side, called the perturbation term, which renders the solution to be a *polynomial*. The perturbation term is expressed in terms of polynomials with free coefficients (the Tau parameters in Ortiz [7]). A norm of these polynomials satisfies a minimum condition; Chebyshev polynomials are often a convenient choice, but other types of polynomials have been used in practise. For a discussion of Chebyshev, Legendre and other types of perturbation terms and their influence on the error of approximation of the Tau method see Namasivayam and Ortiz [4]. The approach of [12], called *operational* by its authors, leads to the coefficients of a Tau bivariate approximate solution of a linear partial differential equation provided its coefficients and the right hand side of the equation are polynomials (or rational polynomials) or polynomials approximation of a given accuracy of the original functions. We shall not discuss here the details of the construction of the Tau approximate solution of such problem; the reader can find a detailed discussion in the paper of Ortiz and Samara [12].

Let $u(x, y)$ be the exact solution of a given nonlinear partial differential equation. We shall assume that nonlinear terms involving $u(x, y)$ (or its derivatives) are decomposed into the product of an approximation of $u(x, y)$ (or its derivatives) and another function. An initial guess is made for such function at the start of the approximation process and the result of stage n is used to activate stage $n+1$. Provided that the process is convergent (see [3]), the elements of a sequence of approximations of $u(x, y)$ is generated, the fixed point of which is precisely the exact solution of the given problem, as the perturbation term in the Tau problem converges to zero. This last statement is used as a control in the computational process to verify that local convergence is taking place. Let us consider a concrete example:

Example 1.

$$-\Delta u(x, y)+\lambda u(x, y)+u^3(x, y)=f(x, y) \tag{1a}$$

in a open domain Ω of $\mathbb{R}^2$, and

$$u(x, y)\,|_{\partial\Omega}=0, \tag{1b}$$

where $\partial\Omega$ is the boundary of Ω. Let us assume throughout this paper that

$$\Omega=\{(x, y)\in\mathbb{R}^2, 0<x, y<1\},$$

then, for

$$f(x, y)= -200x(x-1)-200y(y-1)$$
$$+100\lambda x(x-1)y(y-1)+10^6x^3(x-1)^3y^3(y-1)^3,$$

the exact solution is

$$u(x, y)=100xy(x-1)(y-1).$$

This example has been considered by Temam [14, pp. 154–155], who gives numerical results obtained for it by using the method of fractionary steps. As Temam points out, results obtained with that method are sensitive to the value of λ. He chooses $\lambda = 1$ and $\lambda = 100$. Let us introduce the iterative scheme

$$\begin{cases} -\Delta u_{n+1}(x, y) + \lambda u_{n+1}(x, y) + 3u_n^2(x, y)u_{n+1}(x, y) \\ \quad = f(x, y) + 2u_n^3(x, y) \quad \text{in } \Omega, & (2a) \\ u_{n+1}(x, y)\,|_{\partial\Omega} \equiv 0 \quad \text{where } \partial\Omega \text{ is the boundary of } \Omega; & (2b) \end{cases}$$

for $n \geqslant 1$, where $u_0(x, y)$, the initial guess, is a given function. Clearly (2a) follows from (1a) by using a linearization scheme based on Newton–Kantorovich method; other more (or less) sophisticated schemes are possible (see [13]). Let us take $u_0(x, y)$ to be a polynomial of x and y. Then for $n = 1$ (2a) is a linear partial differential equation with polynomial coefficients: it can be solved with the Tau method of [12]. The solution, $u_2^*(x, y)$, is again a polynomial of a prescribed degree r in x and s in y. Thus, the Tau method can be applied to generate a sequence $\{u_n^*(x, y)\}$ of polynomial approximations of $u(x, y)$, solution of (1), for $n = 1, 2, 3, \ldots$. Let us use a more precise notation:

$$u^*_{n(r,s)}(x, y)$$

to indicate the nth Tau approximation of degree r in x and s in y; let

$$E_{n(r,s)} := \| u - u^*_{n(r,s)} \| = \sup_{(x, y) \in \bar{\Omega}} | u(x, y) - u^*_{n(r,s)}(x, y) |$$

be the maximum absolute error of $u^*_{n(r,s)}(x, y)$ over the domain $\bar{\Omega} = \Omega \cup \partial\Omega$.

In Table 1 we give numerical results obtained with a sequence of Tau approximations of degree $r = s = 4$ for $\lambda = 1$ and for $\lambda = 100$; the initial choice was $u_0(x, y) \equiv 0$, and the iterative process was stopped when the absolute value of two consecutive estimates was below a tolerance parameter $\varepsilon = 10^{-8}$.

Even for such low degree as $r = s = 4$, the numerical results obtained with our Tau approximation compares very favourably with those reported by Temam in [13], where a maximum absolute error of 0.367×10^{-2} (for $\lambda = 100$) was reached using a step $h = \frac{1}{20}$.

Example 2. Let $\mathbb{P}$ be the parabolic operator:

$$\mathbb{P} := \partial/\partial t - \partial^2/\partial x^2,$$

Table 1
Maximum absolute error of Tau approximations of degree (4, 4) of the nonlinear partial differential equation (1).

n	$E_{n(4,4)}$ for $\lambda = 1$	$E_{n(4,4)}$ for $\lambda = 100$
1	0.7137×10	0.1364×10
2	0.2877×10	0.9852×10^{-1}
3	0.6799	0.5106×10^{-3}
4	0.4871×10^{-1}	0.1360×10^{-7}
5	0.2709×10^{-3}	0.6336×10^{-9}
6	0.8541×10^{-8}	
7	0.4954×10^{-9}	

and let us consider the nonlinear parabolic problem

$$\begin{cases} \mathbb{P}u(x, y) - u^2(x, y) = f(x, y) \quad \text{in } \Omega, & \text{(3a)} \\ u(x, 0) = g(x), \quad 0 \leqslant x \leqslant 1, & \\ u(0, y) = u(1, y) = h(y), \quad 0 \leqslant y \leqslant 1. & \text{(3b)} \end{cases}$$

For $f(x, y) = \exp y \sin \pi x(1 + \pi^2 - \exp y \sin \pi x)$; $g(x) = \sin \pi x$, and $h(y) \equiv 0$, the exact solution of (3) is

$$u(x, y) = \exp y \sin \pi x.$$

We shall associate with (3) the iterative scheme

$$\begin{cases} \mathbb{P}u_{n+1}(x, y) - 2u_n(x, y)u_{n+1}(x, y) = \hat{f}(x, y) - u_n^2(x, y), \\ u_{n+1}(x, 0) = g(x), \quad 0 \leqslant x \leqslant 1, \\ u_{n+1}(0, y) = u_{n+1}(1, y) = h(y), \quad 0 \leqslant y \leqslant 1, \end{cases}$$

where $\hat{f}$ stands for a bivariate polynomial approximation of $\hat{f}$. Let us take as an initial guess

$$u_0(x, y) = \sin \pi x.$$

As in the previous example, to be able to apply the Tau method we must take a polynomial initial guess. Let it be a high accuracy polynomial approximation of $\sin \pi x$; such approximation can be obtained by using the software for the Tau method described in [10]. An alternative is to interpolate it at Chebyshev points, which minimize the interpolation error. We have used approximations of degrees $m = 6$, 8 and 10 for the cases (6, 6); (8, 8) and (10, 10) respectively. The approximation error in each of these cases is 0.1332×10^{-4}, 0.1396×10^{-6} and 0.6580×10^{-9} respectively. In Table 2 we report numerical results obtained by using the Tau method for linear partial differential equations with variable coefficients with the polynomial initial guesses indicated before. For the Tau approximations we selected degrees $r = s = 6$, 8 and 10 respectively.

Example 3. Finally, let us consider the hyperbolic operator $\mathbb{H}$:

$$\mathbb{H} := \partial^2/\partial y^2 - \partial^2/\partial x^2,$$

and the nonlinear hyperbolic problem

$$\begin{cases} \mathbb{H}u(x, y) - u^2(x, y) = f(x, y) \quad \text{in } \Omega, & \text{(4a)} \\ u(x, 0) = \partial u(x, y)/\partial y|_{y=0} = g(x), \quad 0 \leqslant x \leqslant 1, & \\ u(0, y) = u(1, y) = h(y), \quad 0 \leqslant y \leqslant 1. & \text{(4b)} \end{cases}$$

Table 2
Maximum absolute error of Tau approximations of degrees (6, 6), (8, 8) and (10, 10) of the nonlinear partial differential equation (3).

n	$E_{n(6,6)}$	$E_{n(8,8)}$	$E_{n(10,10)}$
1	0.2234	0.2236	0.2236
2	0.3322×10^{-2}	0.3271×10^{-2}	0.3272×10^{-2}
3	0.7351×10^{-4}	0.7955×10^{-6}	0.4432×10^{-6}
4	0.7343×10^{-4}	0.4122×10^{-6}	0.4485×10^{-8}
5	0.7343×10^{-4}	0.4122×10^{-6}	0.4485×10^{-8}

Table 3
Maximum absolute error of Tau approximations of degrees (6, 6), (8, 8) and (10, 10) of the nonlinear partial differential equation (4).

n	$E_{n(6,6)}$	$E_{n(8,8)}$	$E_{n(10,10)}$
1	0.1145×10^{-1}	0.1146×10^{-1}	0.1146×10^{-1}
2	0.1475×10^{-3}	0.1640×10^{-5}	0.6157×10^{-6}
3	0.1470×10^{-3}	0.9190×10^{-6}	0.4092×10^{-8}
4	0.1500×10^{-3}	0.1011×10^{-5}	0.4424×10^{-7}
5	0.1470×10^{-3}	0.9190×10^{-6}	0.4462×10^{-7}

For $f(x, y) = \exp y \sin \pi x(1 + \pi^2 - \exp y \sin \pi x)$, $g(x) = \sin \pi x$, and $h(y) \equiv 0$, the exact solution is

$$u(x, y) = \exp y \sin \pi x.$$

We shall use again a linearization scheme based on Newton–Kantorovich method:

$$\begin{cases} \mathbb{H} u_{n+1}(x, y) - 2u_n(x, y)u_{n+1}(x, y) = -u_n^2(x, y) + f(x, y), \\ u_{n+1}(x, 0) = \partial u(x, y)/\partial y\big|_{y=0} = g(x), \quad 0 \leqslant x \leqslant 1, \\ u(0, y) = u(1, y) = h(y), \quad 0 \leqslant y \leqslant 1, \end{cases}$$

with the choices of $\hat{f}$, g and h indicated above. Let us take as an initial guess

$$u_0(x, y) = (1 + y) \sin \pi x.$$

For the construction of the sequence of Tau approximations we take, as before, a polynomial approximation of that initial guess.

In Table 3 we report numerical results obtained by using the Tau method of [12] for $r = s = 6$, 8 and 10. We notice again that local fixed points are attained for different values of the degree (r, s) of the Tau approximation, and that these local fixed points become closer to $u(x, y)$, exact solution of problem (4), as (r, s) increases.

3. Final remarks

In all examples considered the initial guess has simply been a function which satisfies the given boundary conditions, but other choices are possible (see [6]). For results on the error analysis of the Tau method see [2] and [5]; further details on the Tau method are given in [1]. Numerical experiments have been conducted by the authors with: (i) perturbation terms which are not necessarily of Chebyshev type, in particular, Legendre polynomials have been considered (see [6]); (ii) the use of segmentation of the given domain (see Ortiz [8]) into *Tau elements*, and (iii) singular boundary value problems which model the behaviour of cracks in solids. Encouraging results have been obtained, which will be reported elsewhere.

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